'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 3rd Semester Examination, 2021

CC7-MATHEMATICS

RIEMANN INTEGRATION AND SERIES OF FUNCTIONS

Time Allotted: 2 Hours | Full Marks: 60

The figures in the margin indicate full marks.
All symbols are of usual significance.

GROUP-A

1. Answer any ***four*** questions: 3×4 = 12

(a) Use Bonnet's form of second Mean Value Theorem to prove that $\left|\int_a^b \sin x^2 dx\right| \le \frac{1}{a}$; if $0 < a < b < \infty$

(b) Examine the uniform convergence of the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ where $f_n(x) = \frac{x}{1+nx}$, $x \ge 0$.

(c) A function f is defined on $[-2,1]$ by $f(x) = \operatorname{sgn} x$ and $\phi(x) = |x|$. Show that $\int_{-2}^{1} f(x)\,dx = \phi(1) - \phi(-2)$ although $\phi'(x) \ne f(x)$ on $[-2,1]$.

(d) If a function f is continuous on $[a,b]$, $f(x) \ge 0$ on $[-2,1]$ and $\int_a^b f = 0$. Prove that $f \equiv 0$ on $[a,b]$ identically.

(e) Let $f_n(x) = nxe^{-nx^2}$, $n \in \mathbb{N}$ and $0 \le x \le 1$. Determine whether

$$\lim_{n\to\infty} \int_0^{\infty} f_n(x)dx = \int_0^1 \left\{\lim_{n\to\infty} f_n(x)\right\} dx$$

(f) Examine if the trigonometric series $\sum_{n=1}^{\infty}(\sin nx + \cos nx)$ is a Fourier series in $[-\pi, \pi]$.

GROUP-B

Answer any *four* questions 6×4 = 24

2. Represent $f(x)$, where $f(x) = \cos px$, $-\pi \le x \le \pi$ (p is not being an integer) in Fourier series. Deduce that $\frac{\pi}{\sin px} = \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{n+p} + \frac{1}{n-p+1}\right)$. 3+3

3. For each $n \in \mathbb{N}$, let $f_n(x) = nx^2$, $0 \le x \le \frac{1}{n}$ and $f_n(x) = x$, $\frac{1}{n} < x \le 1$. 2+2+2

(i) Show that the sequence $\{f_n\}$ converges to a function f on $[0, 1]$.

(ii) Find M_n where $M_n = \sup\limits_{n \in [0,1]} |f_n(x) - f(x)|$

(iii) Show that $\{f_n\}$ is uniformly convergent on $[0, 1]$.

4. Show that $\int_0^1 \log \Gamma(x) dx$ is convergent and hence find the value of the integral. 6

5. Prove that $\frac{\pi^3}{24\sqrt{2}} < \int_0^{\pi/2} \frac{x^2}{\sin x + \cos x} dx < \frac{\pi^3}{24}$. 6

6. Consider the function f defined on $[0, 1]$ as follows; 4+2

$$f(x) = 0, \text{ when } x \text{ is irrational or zero}$$
$$= \frac{1}{q}, \text{ when } x = \frac{p}{q} \text{ with HCF}(p, q) = 1$$

Examine Riemann integrability of f and find the value of the integral if exists.

7. As an application of Abel's theorem show that $\Sigma a_n \cdot \Sigma b_n = \Sigma c_n$ where Σa_n, Σb_n, Σc_n are convergent infinite series and $c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0$. 6

GROUP-C

Answer any *two* questions 12×2 = 24

8. (a) Test the convergence of $\int_0^1 \frac{dx}{e^x - \cos x}$ 2

(b) Find the value of m for which the integral $\int_0^1 (\log \frac{1}{x})^m dx$ converges. 5

(c) Let $r_1, r_2, r_3 \cdots$ be an enumeration of the set of all rational points in $[0, 1]$ and a sequence of functions $(f_n)_{n \in \mathbb{N}}$ is defined on $[0, 1]$ as follows 5

$$f_n(x) = \partial, \quad x = r_1, r_2, r_3, \cdots r_n$$
$$= 1, \quad x \in [0, 1] \setminus \{r_1, r_2, \cdots r_n\}.$$

Show that the sequence (f_n) is not uniformly convergent on $[0, 1]$.

9. (a) Test the convergence of $\int_0^1 x^{m-1}(1-x)^{n-1} dx$ 7

(b) Show that $\int_2^\infty \frac{\cos x}{\log x}$ is conditionally convergent. 5

10.(a) Let $f(x) = x[x]$, $x \in [0, 4]$ 4

Show that f is integrable on $[0, 4]$ and $\int_0^4 f(x)dx = 17$.

(b) Test the convergence of $\int_0^1 x^x dx$ 4

(c) Using power series expansion of $(1+x^2)^{-1}$ show that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$. 4

11.(a) Show that Fourier series of a function f may not converge to f. 3

(b) Show that the Fourier series $\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{\sin nx}{2n} - \frac{\cos nx}{n^2}\right)$ converges to the periodic function f in $(-\pi, \pi)$ where $f(x) = x^2 + x$, for $-\pi < x < \pi$ and $f(x) = \pi^2$ for $x = \pm\pi$. 9

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